Section 4.2

Rolle's Theorem and MVT

We are currently on a quest to understand as much about a function's behavior as possible using our now well established tool - the derivative. We now look at the next related result, the Mean Value Theorem and its useful, simpler brother Rolle's Theorem.

Theorem 4.2.1 (Rolle's Theorem)**.** Let f be continuous on $[a, b]$ and differentiable on (a, b) such that $f(a) = f(b)$. There is at least one point c in (a, b) such that $f'(c) = 0$.

Hopefully, we can see the connection to the Extreme Value Theorem.

As with all of our Calculus Theorems, there is a reason we require f to be continuous:



Example 4.2.2. Show that $f(x) = x^2 \cos 4x$ has a point with $f'(x) = 0$ on the interval $\left[\frac{\pi}{8}, \frac{3\pi}{8}\right]$.

The Mean Value Theorem says essentially the same as Rolle's Theorem for different values of f'

Theorem 4.2.3 (Mean Value Theorem)**.** If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We can get an intuition for why the MVT works by looking at the graph of a function.

Example 4.2.4. A police helicopter times you driving between 1 mile markers in 48 seconds through a 65 mph zone. A patrol car pulls you over for speeding, but you have been driving 65 (and certainly at any point the patrol car was in view). How is your impending speeding ticket justified?

Consequences of MVT

We have already known that the derivative of a constant function is zero; surprisingly, we haven't known if the only functions whose derivative is zero are constant functions.

Theorem 4.2.5. If f is differentiable and $f'(x) = 0$ for all points on an open interval I, then f is constant on I.

Proof.

Theorem 4.2.6. If two functions have the same derivative $f'(x) = g'(x)$ for all x in an open interval I, then $f(x) - g(x) = C$ on I; that is, f and g differ by a constant.

Proof.